On the representation of orthogonally additive polynomials in ℓ_p.

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ABSTRACT. We present a new proof of a Sundaresan's result which shows that the space of orthogonally additive polynomials $\mathcal{P}_o(^k\ell_p)$ is isometrically isomorphic to $\ell_{p/p-k}$ if $k < p < \infty$ and to ℓ_∞ if $1 \le p \le k$.

1. Introduction

A continuous scalar-valued map P in a Banach space is called a k-homogeneous polynomial if there exists a continuous k-linear form ϕ on X such that $P(x) = \phi(x, \ldots, x)$. The space of k-homogeneous polynomials on X will be denoted by $\mathcal{P}(^kX)$. It is a Banach space with the norm

$$\|P\| = \sup_{\|x\| \le 1} |P(x)|$$

We will denote by $\widehat{\bigotimes}_{\pi,k} X$ and by $\widehat{\bigotimes}_{\pi,s,k} X$ the completed k-fold projective tensor product and the completed k-fold projective symmetric tensor product respectively, where π denotes the projective norm. For $X = \ell_p$, the tensor diagonal is defined to be the closed subspace of $\widehat{\bigotimes}_{\pi,s,k} X$ generated by $e_n \otimes \overbrace{\cdots}^{k} \otimes e_n$. It will be denoted by $D_{k,p}$.

It will be needed the well known result that the dual of the space $\widehat{\bigotimes}_{\pi,s,k} X$ is isometrically isomorphic to the space $\mathcal{P}(^kX)$. For notations and results about homogeneous polynomials, the reader is referred to [**4**] or [**6**], and for the theory of tensor products to [**8**].

We are interested in the subspace of $\mathcal{P}(^kX)$ consisting of all orthogonally additive polynomials. Recall that if X is a Banach lattice, a k-homogeneous polynomial P is said to be orthogonally additive if $P(x + y) = P(x) + P(y)$ whenever x and y are orthogonal or disjoint elements of X (that is $|x| \wedge |y| = 0$). We will consider $X = \ell_p$ as a Banach lattice with its natural order given by $x = (x_n) \le y = (y_n)$ if $x_n \le y_n$. See [**5**] for more information about the theory of Banach lattices.

We give here a new proof of a result of Sundaresan (see [**9**]). This result has been recently generalized to all Banach lattices by Benyamini, Lasalle and Llavona in [**2**]. There are also independent proofs for the case of $X = C(K)$, see [**3**] and [**7**]. The reason to present this new proof is that, in our opinion, it is much simpler in the sense that there is no need of hard tools such us the representation of orthogonally additive functionals using Caratheodory functions, as in [**9**], or the Kakutani representation theorem for Banach lattices used in [**2**]. This new proof also makes more evident the underlying ideas.

2. The tensor diagonal

In this section we generalize Example 2.23 in [**8**] to give a description of the tensor diagonal in $\widehat{\bigotimes}_{\pi,s,k} \ell_p$. The main technique was a Rademacher averaging ([**8**], Lema 2.22). Its generalization requires the k-Rademacher generalized functions introduced by Aron and Globevnik in [**1**]:

DEFINITION 2.1 ([**1**]). Fix $k \in \mathbb{N}$, $k \geq 2$ and let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_k$ denote the nth roots of unity. Let $r_1 : [0,1] \longrightarrow \mathbb{C}$ be the step function taking the value α_j on $(j - 1/k, j/k)$ for $j = 1, \ldots, n$. Assuming that r_{n-1} has been defined, define r_n in the following way: fix any of the k_{n-1} sub-intervals I of $[0,1]$ used in the definition of r_{n-1}. Divide I into k equal intervals $I_1, \ldots, I_k$ and set $r_n(t) = \alpha_j$ if $t \in I_j$.

We will also need the following

LEMMA 2.2 ([**1**]). *For each $k = 2, 3, \ldots$ the associated functions r_n satisfy the following properties:*

- $|r_n(t)| = 1$ *for all $n \in \mathbb{N}$ and all $t \in [0,1]$.*
- *For any choice of $n_1, \ldots, n_k$*

$$\int_0^1 r_{n_1}(t) \cdots r_{n_k}(t) = \left\{ \begin{array}{l} 1 \text{ if } n_1 = \cdots = n_k \\ 0 \text{ otherwise.} \end{array} \right.$$

The next lemma is a generalization of Lemma 2.22 in [**8**]:

LEMMA 2.3 (Rademacher Averaging). *Let $X_1, \ldots X_k$ vector spaces and let $x_{1,1}, \ldots, x_{1,n} \in X_1, \ldots,$ and $x_{k,1}, \ldots, x_{k,n} \in X_k$. Then*

$$\sum_{i=1}^n x_{1,i} \otimes \cdots \otimes x_{k,i} = \int_0^1 \left(\sum_{i=1}^n r_i(t) x_{1,i} \right) \otimes \cdots \otimes \left(\sum_{i=1}^n r_i(t) x_{k,i} \right) \mathrm{d}t$$

PROOF. Just expand the integral and use the second property of Lemma 2.2. □

THEOREM 2.4. *Let $1 \leq p < \infty$. The tensor diagonal $D_{k,p}$ in $\widehat{\bigotimes}_{\pi,s,k} \ell_p$ is isometrically isomorphic to $\ell_{p/k}$ if $k < p < \infty$ and to ℓ_1 if $1 \leq p \leq k$.*

PROOF.

(1) $k < p < \infty$.
   Let $u = \sum_{i=0}^n a_i e_i \otimes \cdots \otimes e_i \in D_{k,p}$. Using Lemma 2.3, we write

$$u = \int_0^1 \left(\sum_{i=1}^n sign(a_i) |a_i|^{1/k} r_i(t) e_i \right) \otimes \cdots \otimes \left(\sum_{i=1}^n |a_i|^{1/k} r_i(t) e_i \right) \mathrm{d}t$$

and, like in [**8**] (pages 34-35) we get:

$$\pi(u) \leq \sup_{0 \leq t \leq 1} \left\| \sum_{i=1}^{n} sign(a_i)|a_i|^{1/k}r_i(t)e_i \right\|_p \cdots \left\| \sum_{i=1}^{n} |a_i|^{1/k}r_i(t)e_i \right\|_p =$$

$$= \left(\sum_{i=1}^{n} |a_i|^{p/k} \right)^{k/p} = \|(a_i)\|_{p/k}$$

To prove the identity, define a k-linear form on ℓ_p by $B(x_1, \ldots, x_n) = \sum b_i x_{1,i} \cdots x_{k,i}$ where $x_j = (x_{j,n})$ and $b_i = sign(a_i)|a_i|^{p/k-1}$. Using Hölder's inequality, it is easy to see that $\|B\| \leq (\sum_{i=1}^{n} |a_i|^{p/k})^{1-p/k}$ and then

$$\sum_{i=1}^{n} |a_i|^{p/k} = |\langle u, B \rangle| \leq \pi(u)(\sum_{i=1}^{n} |a_i|^{p/k})^{1-p/k}.$$

Hence $\|(a_i)\|_{p/k} \leq \pi(u)$ and therefore $D_{k,p}$ is isometrically isomorphic to $\ell_{p/k}$.

(2) $1 \leq p \leq k$.
   Let be $u = \sum_{i=0}^{n} a_i e_i \otimes \cdots \otimes e_i \in D_{k,p}$, then $\pi(u) \leq \sum_{i=0}^{n} |a_i|$. Reciprocally, define $B(x_1, \ldots, x_n) = \sum_{i=0}^{n} sign(a_i)x_{1,i} \cdots x_{k,i}$, we have that $|B(x_1, \ldots, x_n)| \leq \|x_1\|_k \cdots \|x_k\|_k \leq \|x_1\|_p \cdots \|x_k\|_p$ and so $\|B\| \leq 1$. Then

$$\pi(u) \geq \langle u, B \rangle = \sum_{i=1}^{\infty} |a_i|$$

and we are done.

$\square$

REMARK 1. Definition 2.1 gives the classical Rademacher functions for the case $k = 2$ and these results are those in [**8**], (Example 2.23, page 34).

3. The main result

Our proof is based in the fact that the orthogonally additive polynomials are isometrically isomorphic to the dual of the tensor diagonal $D_{k,p}$. We need a previous lemma:

LEMMA 3.1. *Let $1 \leq p < \infty$. The dual of the tensor diagonal $D_{k,p}^*$ is isometrically isomorphic to ℓ_∞ if $1 \leq p \leq k$ and to $\ell_{p/p-k}$ if $k < p < \infty$ in the sense that for every $F \in D_{k,p}^*$, $(F(e_i \otimes \cdots \otimes e_i))$ is in ℓ_∞ for the first case and in $\ell_{p/p-k}$ for the second.*

PROOF. The proof is standard, observe that $\ell_{p/p-k}$ is the dual of $\ell_{p/k}$ and carry on the same proof of $\ell_q^* = \ell_{q'}$ for $1/q + 1/q' = 1$ with the identification of the projective norm shown in Theorem 2.4. $\square$

We are now ready to prove the theorem:

THEOREM 3.2. *Let $1 \leq p < \infty$. The space of orthogonally additive, k-homogeneous polynomials $\mathcal{P}_o(^k\ell_p)$ is isometrically isomorphic to ℓ_∞ for $1 \leq p \leq k$ and to $\ell_{p/p-k}$ for $k < p < \infty$.*

PROOF. The proof consists on showing that $\mathcal{P}_o(^k\ell_p)$ is isometrically isomorphic to $D^*_{k,p}$. We will suppose that $k < p < \infty$. The other case is analogous.

Let $F \in D^*_{k,p}$, the correspondence is established by associating F to the polynomial $P(x) = \widetilde{F}(x \otimes \cdots \otimes x)$ where $\widetilde{F} \in (\widehat{\bigotimes}_{\pi,k,s}\ell_p)^*$ is defined by

$$\widetilde{F}(e_{n_1} \otimes \cdots \otimes e_{n_k}) = \left\{ \begin{array}{l} F(e_{n_1} \otimes \cdots \otimes e_{n_k}) \text{ if } n_1 = \cdots = n_k \\ 0 \text{ otherwise} \end{array} \right.$$

To see that $\widetilde{F}$ is well defined let $x = \sum x_n e_n \in \ell_p$, then

$$\begin{aligned} |\widetilde{F}(x \otimes \cdots \otimes x)| &\leq \sum |x_{n_1} \cdots x_{n_k} \widetilde{F}(e_{n_1} \otimes \cdots \otimes e_{n_k})| = \sum |x_n^k F(e_n \otimes \cdots \otimes e_n)| \\ &\leq \|(x_n^k)\|_{p/k} \|(F(e_n \otimes \cdots \otimes e_n))\|_{p/p-k} \leq \|F\|\|x\|_p^k = \\ &= \|F\|\pi(x \otimes \cdots \otimes x) \end{aligned}$$

hence $\widetilde{F}$ is continuous and $\|\widetilde{F}\| \leq \|F\|$, then we have the equality since $\widetilde{F}$ was an extension of F.

To see that $P(x) = \widetilde{F}(x \otimes \cdots \otimes x)$ is orthogonally additive, note that is enough to check that the k-linear symmetric form ϕ associated to P, verifies that $\phi(e_{n_1}, \ldots, e_{n_k})$ is zero whenever at least two of its entries will be different and this is true by definition of $\widetilde{F}$.

Finally as the correspondence between polynomials and the dual of the symmetric tensor product is an isometric isomorphism, $\|P\| = \|\widetilde{F}\| = \|F\|$ which completes the proof. □

As it has been shown, if $P(x) = \phi(x, \ldots, x)$ is an orthogonally additive polynomial on ℓ_p, the essential information of P is contained in the sequence $(\phi(e_n, \ldots, e_n))$. There is another possible proof of Theorem 3.2 using a result by Zalduendo (see Corollary 1 of [10]):

LEMMA 3.3 ([10]). *Let $k < p$ and let ϕ a continuous k-linear form on ℓ_p. Then*

$$(\phi(e_n, \ldots, e_n)) \in \ell_{p/p-k}.$$

From this result, it can be shown as well that the space of orthogonally additive polynomials is isometrically isomorphic to $\ell_{p/p-k}$ if $k < p$. We don't repeat the proof since the ideas are essentially the same, however the proof presented here is self-contained.

REMARK 2. This method will also be valid for $1 \leq p \leq k$ since in this case, trivially $(\phi(e_n, \ldots, e_n)) \in \ell_\infty$. It is shown in [10] that this is the best characterization.

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